The logo of the University of Delhi is a circular emblem. It features a central shield with a yellow and orange background, flanked by two green floral motifs. Below the shield is a white elephant standing on a green base. The entire emblem is surrounded by a light blue border containing the text 'UNIVERSITY OF DELHI' at the top and 'विद्या ऽ वृत्तिः सत्यम्' at the bottom.

**Subject: Maths, Differential Equation and Mathematical Modeling-I**  
**Lesson: Mathematical Models**  
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## 1. Learning Outcomes:

After studying this lesson, reader should be able to

- derive one - dimensional wave equation;
- derive two - dimensional wave equation;
- derive one - dimensional heat conduction equation;
- derive Laplace's equation;
- state conservation law and derive Burgers equation;

## 2. Introduction:

In our day to day life, we face problems arising from different disciplines - physics, chemistry, biology, sociology, management, finance etc. Mathematical modelling consists of simplifying the real world problems and representing them in mathematical language, as well as solving the mathematical problems and interpretation of these solutions in the real world language. At some point of time, while studying mathematics, we must have solved problems around us. So far the physical systems have been primarily studied by ordinary differential equations. Now we are interested in all those phenomena (or physical process) that requires partial derivatives in the describing equation. Partial differential equations are frequently used to formulate the laws of nature and to study the physical, chemical and biological models. In this chapter, we will study the partial differential equations representing the mathematical models of physical problems in detail.

### 3. Mathematical modelling of the vibrating string:

The motion of vibration of a tightly stretched, flexible string is the most interesting and important problems in applied mathematics and mathematical physics. It was modelled approximately 250 years ago and still widely used as an excellent introductory example.

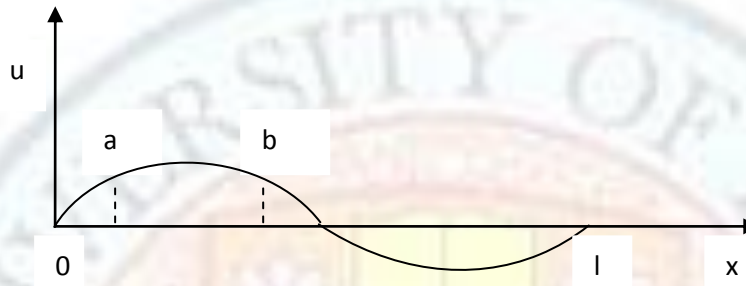


Fig 1(a) Deformed, flexible string of length  $l$  at an instant  $t$ .

Let the length of the stretched string, which is fastened at each end, is  $l$ . We wish to get a describing equation for the deflection  $u$  of the string for any position  $x$  and for any time  $t$ . Consider a differential element of the string at a particular instant enlarged in Fig 1(b).

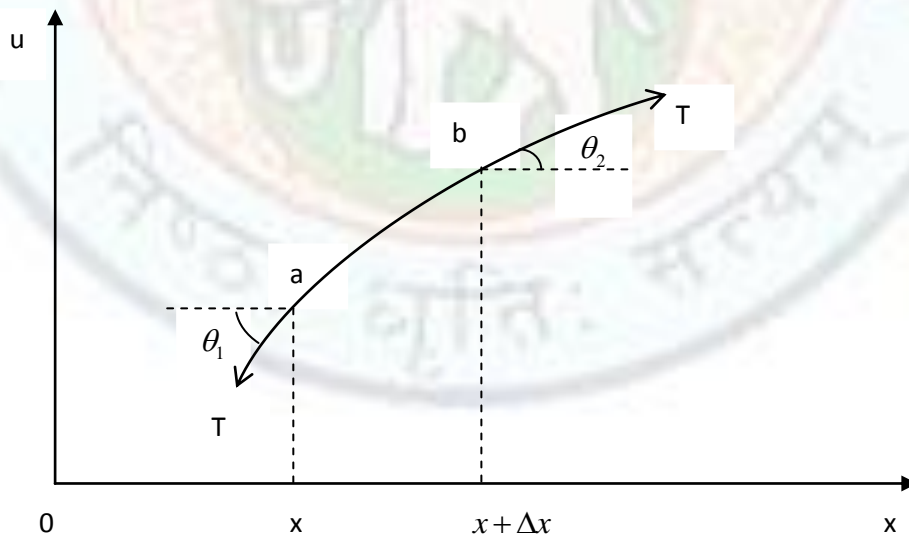


Fig 1(b) Small element of vertically displaced string

## Mathematical Models

We make the following assumptions to obtain a simple equation describing the vibration of the stretched string.

1. The string is elastic and flexible and therefore offers no resistance to bending so that no shearing force exists on a surface normal to the string.
2. The tension is so large that the weight of the string is negligible.
3. There is no elongation of a single segment of the string and hence the tension is constant ( Hooke's Law ).
4. The slope of deflection curve is small. So if  $\theta$  is the inclination angle of the tangent to the deflection curve then we can replace  $\sin\theta$  by  $\tan\theta$ .
5. The deflection is negligible as compared to the string's length so that the resulting change in length of the string has no effect upon the tension.
6. There is only pure transverse vibration, i.e., the motion takes place entirely in one plane and every particle moves at right angles to the equilibrium position of the string in this plane.

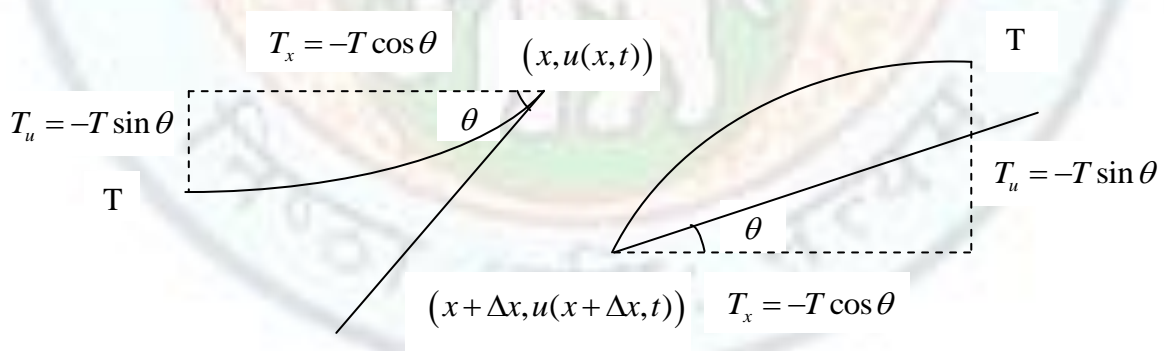


Fig 1(c) Vector representation of tension at x

Let the tension at the end points is  $T$  as shown in Fig 1(b). The forces acting, in the vertical direction, on the element of the string are

$$T \sin \theta_2 - T \sin \theta_1$$

using the Newton's second law of motion we know that  $F = m \mathbf{a}$ , and therefore

$$T \sin \theta_2 - T \sin \theta_1 = \rho \delta s u_{tt} \quad (1)$$

Where  $\delta s$  is the arc length,  $\rho$  is the line density of the string and therefore

$$m = \rho \delta s \text{ and } a = u_{tt}.$$

By assumption 5, we have

$$\delta s \approx \delta x$$

$$\sin \theta_2 \approx \tan \theta_2 = u_x(x + \delta x, t) \text{ and } \sin \theta_1 \approx \tan \theta_1 = u_x(x, t) \text{ at time } t.$$

Equation (1) can then be written as

$$T[u_x(x + \delta x, t) - u_x(x, t)] = \rho \delta x u_{tt}$$

or, equivalently,

$$\frac{1}{\delta x}[u_x(x + \delta x, t) - u_x(x, t)] = \frac{\rho}{T} u_{tt} \quad (2)$$

Now, we let  $\delta x \rightarrow 0$  which also implies  $\delta u \rightarrow 0$ , then by definition,

$$\lim_{\delta x \rightarrow 0} \frac{1}{\delta x}[u_x(x + \delta x, t) - u_x(x, t)] = u_{xx}$$

and equation (2) becomes

$$u_{tt} = c^2 u_{xx}, \quad (3)$$

where

$$c^2 = \frac{T}{\rho} \quad (4)$$

(3) is known **one - dimensional wave equation** and **c** denotes the **wave speed**. It is a transverse wave and it moves normal to the vibrating string.

If some external force **f** per unit length is acting on the string then equation (3) can be given by

$$u_{tt} = c^2 u_{xx} + F, \quad F = \frac{f}{\rho} \quad (5)$$

where  $\mathbf{f}$  can be gravitation, impulse along the string at different values of time, resistance, pressure etc.

This equation is hyperbolic. We will consider the initial and boundary conditions in detail when the solution is presented.

**Value Additions:**

A partial differential equation of second order i.e.,

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where A, B, C, D, E, F and G are constants, is said to be

- (i) Elliptic if  $B^2 - 4AC < 0$
- (ii) Parabolic if  $B^2 - 4AC = 0$
- (iii) Hyperbolic if  $B^2 - 4AC > 0$ .

**Example 1:** Prove that  $u_{tt} = c^2 u_{xx} - g$ , is the equation of motion of a long string where  $g$  is the gravitational acceleration.

**Solutions:** Total force acting on the portion of the string of length  $\delta x$  in the vertical direction is

$$T \sin \theta_2 - T \sin \theta_1 - \rho \delta x g .$$

Therefore the equation of motion is given by

$$T (\sin \theta_2 - \sin \theta_1) - \rho \delta x g = \rho \delta x u_{tt} .$$

Also, by assumption 5, we have

$$\sin \theta_1 \approx \tan \theta_1 = u_x(x, t) \text{ and } \sin \theta_2 \approx \tan \theta_2 = u_x(x + \delta x, t)$$

This implies,

$$\frac{T}{\rho} \left[ \frac{u_x(x + \delta x, t) - u_x(x, t)}{\delta x} \right] - g = u_{tt}$$

As  $\delta x \rightarrow 0$ , we get

## Mathematical Models

$$u_{tt} = c^2 u_{xx} - g, \quad \text{where } c^2 = \frac{T}{\rho}.$$

**Example 2: (a)** Show that the equation  $u_{tt} + \alpha u_t = c^2 u_{xx}$ , represents the *damped wave equation* of a string where the damping force is directly proportional to the velocity and  $\alpha$  is a constant.

**(b)** Show that for a restoring force which is directly proportional to the displacement of a string the resulting equation is

$$u_{tt} + \alpha u_t + bu = c^2 u_{xx},$$

Where  $b$  is a constant.

### Solutions:

**(a)** Damping force  $F_1 \propto u_t \Rightarrow F_1 = \mathbf{a} u_t$ , where  $\mathbf{a}$  is a constant.

Total force in vertical direction is

$$T \sin \theta_2 - T \sin \theta_1 - F_1 \rho \delta x$$

Therefore the equation of motion is

$$T (\sin \theta_2 - \sin \theta_1) - F_1 \rho \delta x = \rho \delta x u_{tt}$$

Also, by assumption 5, we have

$$\sin \theta_1 \approx \tan \theta_1 = u_x(x, t) \quad \text{and} \quad \sin \theta_2 \approx \tan \theta_2 = u_x(x + \delta x, t)$$

This implies,

$$\frac{T}{\rho} \left[ \frac{u_x(x + \delta x, t) - u_x(x, t)}{\delta x} \right] - F_1 = u_{tt}$$

As  $\delta x \rightarrow 0$ , we get

$$u_{tt} = c^2 u_{xx} - F_1 \quad \text{where } c^2 = \frac{T}{\rho}.$$

$$c^2 u_{xx} = u_{tt} + \mathbf{a} u_t \quad (\text{since } F_1 = \mathbf{a} u_t)$$

**(b)** Restoring force,  $F_2 \propto u \Rightarrow F_2 = \mathbf{b} u$  where  $\mathbf{b}$  is a constant.

Equation of motion is given by

$$T (\sin \theta_2 - \sin \theta_1) - F_1 \rho \delta x - F_2 \rho \delta x = \rho \delta x u_{tt}$$



Proceeding in the similar manner as in part (a), we get

$$u_{tt} + \alpha u_t + \beta u = c^2 u_{xx}.$$

### 3.1. Mathematical modelling of The Vibrating Membrane:

A stretched vibrating membrane is an extension into a second space dimension of the vibrating string problem. It occurs in many problems in mathematics and physics. We wish to derive a describing equation for the deflection  $u$  of the membrane for any position  $(x, y)$  and for any time  $t$ . To get a simple equation describing the vibration of the stretched membrane, the following assumptions are considered:

1. membrane is elastic and flexible, therefore offers no resistance to bending so shearing stresses are absent.
2. No elongation of a single segment of the membrane and hence the tension is constant ( Hooke's Law ).

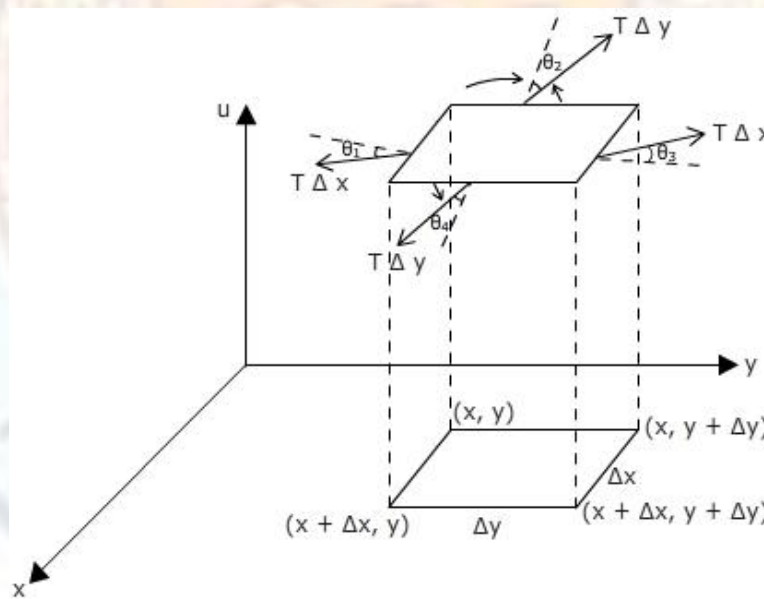


Fig 2. Element of vertically displaced membrane

3. The tension per unit length is so large that the weight of the membrane is negligible.
4. Only pure transverse vibration will be there.

5. The deflection is negligible as compared to the minimal diameter of the membrane so that the resulting change in diameter of the membrane has no effect upon the tension.
6. The slope of deflection surface is small.

Let us take a small element of the membrane. By assumptions 4 and 5, the area of the element is  $\delta x \delta y$  (Approx.). Let  $T$  be the tensile force per unit length and  $T \delta x$  and  $T \delta y$  are the forces acting on the sides of the element. Then the forces which are acting, in the vertical direction, on the element of the membrane are

$$T \delta x \sin \theta_2 - T \delta x \sin \theta_1 + T \delta y \sin \theta_4 - T \delta y \sin \theta_3$$

Also, by assumption 5, we have

$$\begin{aligned} \sin \theta_1 &\approx \tan \theta_1 = u_y(x_1, y) \\ \sin \theta_2 &\approx \tan \theta_2 = u_y(x_2, y + \delta y) \\ \sin \theta_3 &\approx \tan \theta_3 = u_x(x, y_1) \\ \sin \theta_4 &\approx \tan \theta_4 = u_x(x + \delta x, y_2) \end{aligned} \quad (6)$$

Where  $x_1, x_2, y_1$  and  $y_2$  are the values of  $x$  and  $y$  between  $x$  &  $x + \delta x$  and  $y$  &  $y + \delta y$  respectively.

Applying Newton's second law of motion to an element of the membrane as shown in Fig. 2., we get

$$T \delta x (\tan \theta_2 - \tan \theta_1) + T \delta y (\tan \theta_4 - \tan \theta_3) = \rho \delta x \delta y u_{tt} \quad (7)$$

where  $\delta x \delta y$  is the area of the element,  $\rho$  is the mass per unit area and  $u_{tt}$  is the acceleration, calculated at some point in any region of consideration.

Substituting values from equation (6) into equation (7), we get

$$T \delta x [u_y(x_2, y + \delta y) - u_y(x_1, y)] + T \delta y [u_x(x + \delta x, y_2) - u_x(x, y_1)] = \rho \delta x \delta y u_{tt} \quad (8)$$

Dividing equation (8) by  $\rho \delta x \delta y$ , we get

$$\frac{T}{\rho} \left[ \frac{u_y(x_2, y + \delta y) - u_y(x_1, y)}{\delta y} + \frac{u_x(x + \delta x, y_2) - u_x(x, y_1)}{\delta x} \right] = u_{tt}$$

Now, taking the limit as  $\delta x \rightarrow 0$  and  $\delta y \rightarrow 0$ , we arrive at

$$u_{tt} = c^2 (u_{xx} + u_{yy}), \quad (9)$$

where

$$c^2 = \frac{T}{\rho} \quad (10)$$

(9) represents the **two - dimensional wave equation** and **c** denotes the wave speed.

**Value Addition:**

Note: If some external force **f** per unit length is acting on the membrane then two-dimensional wave equation can be given by

$$u_{tt} = c^2 (u_{xx} + u_{yy}) + F, \quad F = \frac{f}{\rho} \quad (11)$$

**4. Conduction of Heat in Solid:**

**4.1. Derivation of heat equation:**

Let S be the closed surface that bounds an arbitrary domain D. Let T (x, y, z, t) is the temperature at each point (x, y, z) in a solid body D at any time t. By Fourier’s law, we know that the flow rate is directly proportional to the gradient of temperature. Therefore, the velocity of the heat flow in an **isotropic** body is given by

$$\mathbf{v} \propto \nabla T \Rightarrow \mathbf{v} = -k \nabla T \quad (12)$$

where **k** (called thermal conductivity) is a constant.

Since **v** is the velocity of heat flow, amount of heat leaving D per unit time is  $\iint_S \mathbf{v} \cdot \mathbf{N} ds$  and by using Gauss Divergence theorem, we get,

$$\iint_S \mathbf{v} \cdot \mathbf{N} ds = \iiint_D \text{div}(-k \nabla T) dv = \iiint_D (-k \nabla \cdot \nabla T) dv = \iiint_D -k \nabla^2 T dv \quad (13)$$

Since this is the amount of heat leaving D per unit time, it must be equal to the heat integral (by definition). Thus,

## Mathematical Models

$$\iiint_D -k \nabla^2 T \, dv = \iiint_D -\sigma \delta \frac{\partial T}{\partial t} \, dv \Rightarrow -k \nabla^2 T = -\sigma \delta \frac{\partial T}{\partial t}$$

This implies,

$$\frac{\partial T}{\partial t} = \frac{k}{\sigma \delta} \nabla^2 T$$

Or,

$$\frac{\partial T}{\partial t} = K \nabla^2 T \quad \text{where } K = \frac{k}{\sigma \delta} \quad (14)$$

Equation (14) is known as **one - dimensional heat equation**. The above equation is parabolic and will be solved in the subsequent chapters.

### Value Addition:

**Isotropic:** A body is said to be isotropic with respect to some property (Pressure, density etc.) if that property is the same in all direction at a point.

**Note:** The total heat contained in a body D with uniform density  $\delta$  and specific heat  $\sigma$  is  $\iiint_D \sigma \delta T \, dv$  where T is the temperature. Thus, the amount of heat leaving D per unit time is given by the derivation

$$-\frac{\partial}{\partial t} \left[ \iiint_D \sigma \delta T \, dv \right] = \iiint_D -\sigma \delta \frac{\partial T}{\partial t} \, dv \quad (15)$$

**Example 3:** Putting  $U = e^{ikt} u$  in the wave equation  $U_{tt} = \nabla^2 U$  and also putting  $U = e^{-k^2 t} u$  in the heat equation  $U_t = \nabla^2 U$ , prove that  $u(x, y, z)$  satisfies the equation

$$\nabla^2 u + k^2 u = 0 \text{ (Helmholtz equation)}$$

**Solutions:** Putting  $U = e^{ikt} u$  in the wave equation  $U_{tt} = \nabla^2 U$ , we get,

$$\nabla^2 U = e^{ikt} \nabla^2 u$$

Also,

$$U_t = ik e^{ikt} u$$

$$U_{tt} = i^2 k^2 e^{ikt} u = -k^2 e^{ikt} u$$

$$\therefore U_{tt} = \nabla^2 U \Rightarrow -k^2 e^{ikt} u = e^{ikt} \nabla^2 u$$

$$\Rightarrow \nabla^2 u + k^2 u = 0$$

Again, putting  $U = e^{-k^2 t} u$  in the heat equation  $U_t = \nabla^2 U$ , we get,

$$\nabla^2 U = e^{-k^2 t} \nabla^2 u$$

$$U_t = -k^2 e^{-k^2 t} u$$

$$-k^2 e^{-k^2 t} u = e^{-k^2 t} \nabla^2 u$$

and therefore,

$$\nabla^2 u + k^2 u = 0$$

## 5. The Gravitational Potential:

There are a number of physical situation that are modelled by Laplace's equation. We choose the force of attraction of particles to demonstrate its derivation. Suppose that the two particles P and Q having masses  $m$  and  $M$  respectively are separated by a distance  $r$ . The Newton's gravitational law states that the attraction force between the particles is directly proportional to the product of  $m$ ,  $M$  and  $1/r^2$ . So we have,

$$F \propto \frac{mM}{r^2}$$

Consequently,

$$F = G \frac{mM}{r^2} \tag{16}$$

where  $G$  is the gravitational constant.

## Mathematical Models

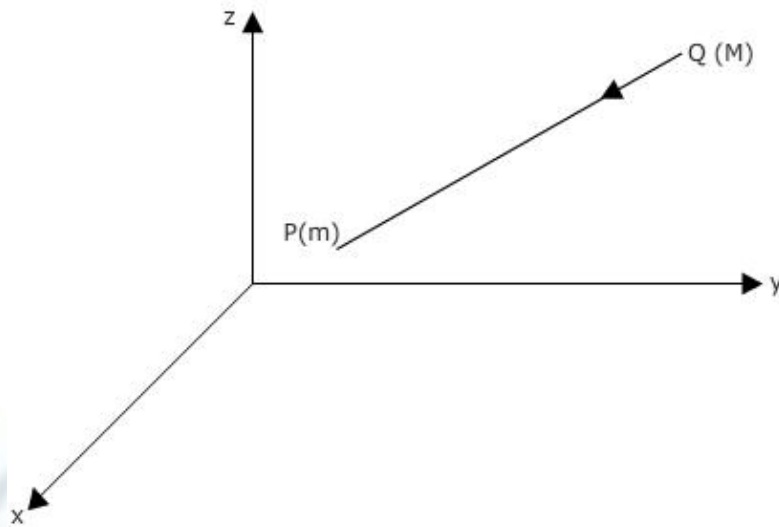


Fig 3 Two particles separated by a distance  $r$

Choosing (Taking)  $G = 1$ , we have,

$$F = \frac{mM}{r^2} \quad (17)$$

If  $\vec{r} = \overrightarrow{PQ}$ , then force per unit mass ( $M = 1$ ) at  $Q$  due to mass ( $m$ ) at  $P$  is given by

$$F = -\frac{m}{r^2} \left( \frac{\vec{r}}{r} \right) = -\frac{m\vec{r}}{r^3} = \nabla \left( \frac{m}{r} \right) \quad (18)$$

Equation (18) is known as the intensity of gravitational field force.

Let us consider a particle of unit mass ( $M = 1$ ) moves in the influence of the attraction of mass  $m$  at  $P$  from infinity up to  $Q$ . Then the work done by the force  $F$  is given by

$$W = \int_{\infty}^r F dr = \int_{\infty}^r \nabla \left( \frac{m}{r} \right) dr = \frac{m}{r} \quad (19)$$

Equation (18) is known as the potential at  $Q$  because of the particle at  $P$ , and is denoted by

$$V = -\frac{m}{r} \quad (20)$$

so that the force intensity at P is given by

$$F = \nabla \left( \frac{m}{r} \right) = -\nabla V \quad (21)$$

Let  $r_1, r_2, \dots, r_n$  be the distances of the masses  $m_1, m_2, \dots, m_n$  respectively from Q. Then the attraction force per unit mass due to the system at Q is given by

$$F = \sum_{i=1}^n \nabla \frac{m_i}{r_i} = -\nabla \sum_{i=1}^n \frac{m_i}{r_i} \quad (22)$$

The work done by the force acting on a particle of unit mass is

$$\int_{\infty}^r F dr = \sum_{i=1}^n \frac{m_i}{r_i} = -V \quad (23)$$

Thus the potential satisfies the equation

$$\nabla^2 V = -\nabla^2 \sum_{i=1}^n \frac{m_i}{r_i} = -\sum_{i=1}^n \nabla^2 \left( \frac{m_i}{r_i} \right) = 0, \quad r_i \neq 0 \quad (24)$$

For continuous distribution of mass in some volume R, the potential  $u$  is defined by

$$u(x, y, z) = k \iiint_R \frac{1}{r} \rho dX dY dZ \quad (25)$$

where  $k$  is a positive constant,  $\rho$  is the density of mass at  $Q(X, Y, Z)$  and  $r = \sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2}$  is the distance between two points.

## Mathematical Models

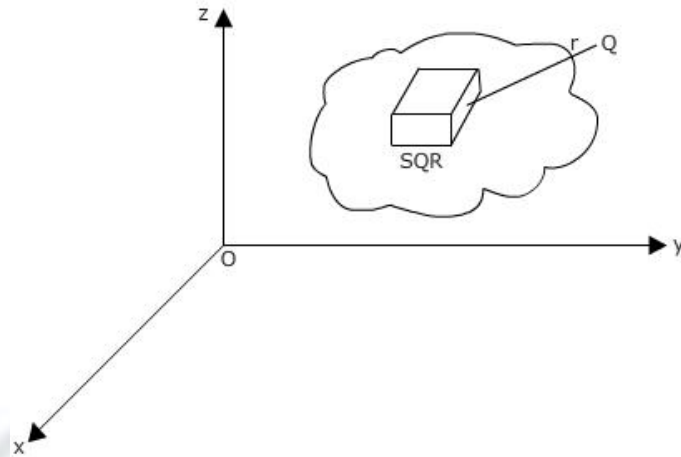


Fig 4 Continuous Mass Distribution

Assuming the resulting function that we derive are continuous. We have,

$$u_x(x, y, z) = -k \iiint_R \frac{\rho(x-X)}{r} dX dY dZ,$$

$$u_{xx}(x, y, z) = -k \iiint_R \left[ \frac{\rho}{r^3} - \frac{3\rho(x-X)^2}{r^5} \right] dX dY dZ$$

Similarly,

$$u_{yy}(x, y, z) = -k \iiint_R \left[ \frac{\rho}{r^3} - \frac{3\rho(y-Y)^2}{r^5} \right] dX dY dZ$$

$$\text{and } u_{zz}(x, y, z) = -k \iiint_R \left[ \frac{\rho}{r^3} - \frac{3\rho(z-Z)^2}{r^5} \right] dX dY dZ$$

Adding these results, we get

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$$

The above equation is known as the Laplace equation or a potential equation and its solutions are called potential function or harmonic function.



**Value Addition:**

**Laplacian Operator:** The operator  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$  is called Laplacian Operator. It is widely used in many physical phenomenon.

**Example 4:** Determine the potential at Q when it is inside the body, (in the derivation of the Laplace equation, considering the potential at Q which is outside the body is ascertained) and show that it satisfies the equation

$$\nabla^2 u = -4\pi\rho, \text{ ( Poisson equation)}$$

where  $\rho$  is the density of the body.

**Solution:** We know that the flux is given by the equation

$$\int_S \vec{F} dS = 4\pi \int_V \rho dV, \tag{26}$$

Also, by the divergence theorem,

$$\int_S \vec{F} dS = \int_V \nabla \cdot \vec{F} dV = \int_V \nabla \cdot (-\nabla V) dV = -\int_V \nabla^2 V dV \tag{27}$$

( $\because \vec{F} = -\nabla V$  force is the gradient of P.E. with reversed sign )

By (26) & (27), we get,

$$\begin{aligned} 4\pi \int_V \rho dV &= -\int_V \nabla^2 V dV \\ \Rightarrow \int_V (4\pi\rho + \nabla^2 V) dV &= 0 \end{aligned}$$

If the integrand is continuous,

$$4\pi\rho + \nabla^2 V = 0 \Rightarrow \nabla^2 V = -4\pi\rho$$

## 6. Conservation Law:

**Statement:** The flux of a material across the closed bounded surface  $S$  of the domain is equal to the rate of change of the total amount of that material contained in a fixed volume  $V$ .

Let  $\rho(\mathbf{x}, t)$  be the density of the material and  $q(\mathbf{x}, t)$  is the flux vector, then by the mass conservation law, we have

$$\frac{d}{dt} \int_V \rho dV = - \int_S (\vec{q} \cdot \vec{n}) dS \quad (28)$$

where  $dS$  is the surface element and  $dV$  is the volume element of the boundary surface  $S$ . The outward unit normal vector to  $S$  is denoted by  $\vec{n}$ .

By Gauss divergence theorem, we know that

$$\int_S (\vec{q} \cdot \vec{n}) dS = \int_V \text{div } \vec{q} dV$$

From, eqn. (28),

$$\begin{aligned} \int_V \frac{\partial \rho}{\partial t} dV &= - \int_S (\vec{q} \cdot \vec{n}) dS = - \int_V \text{div } q dV \\ \Rightarrow \int_V \left( \frac{\partial \rho}{\partial t} + \text{div } q \right) dV &= 0 \end{aligned} \quad (29)$$

This result vanishes everywhere in the domain for continuous integrand and is valid for arbitrary volume  $V$ . Therefore, we get a *differential form* of the conservation law given by

$$\frac{\partial \rho}{\partial t} + \text{div } q = 0 \quad (30)$$

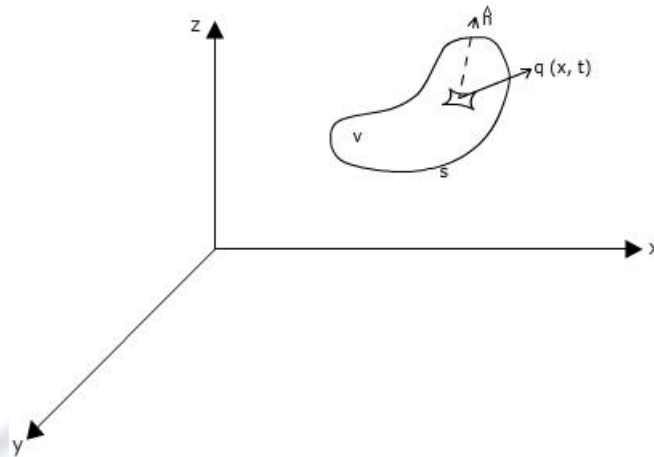


Fig 5: Volume  $V$  and the surface element  $dS$  with the outward unit normal vector  $\mathbf{n}$  of a closed domain bounded by the surface  $S$

The one – dimensional form of eqn. (28) is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (31)$$

**Value Additions:**

Fundamental principle of conservation law (conservation of momentum, energy and angular momentum) have far-reaching implications as symmetries of nature which is not violated and serves as a strong constraint on any theory in any branch of science.

**7. The Burgers Equation:**

If the solution of above equation (2.31) is discontinuous or for any shock waves, we consider  $q = Q(\rho)$  and allow a jump discontinuity for  $\rho$  and  $q$ . Let us assume  $q$  as a function of density gradient  $\rho_x$  as well as  $\rho$ . For a simple model,

$$q = Q(\rho) - v \rho_x \quad (32)$$

where  $v$  is a constant(positive).

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Differentiating equation (30) partially with respect to  $x$ , we get,

$$\frac{\partial q}{\partial t} = Q'(\rho)\rho_x - \nu\rho_{xx} \quad (33)$$

Substituting equation (31) into equation (29), we get,

$$\rho_t + C(\rho)\rho_x = \nu\rho_{xx} \quad (34)$$

where

$$C(\rho) = Q'(\rho) \Rightarrow C'(\rho) = Q''(\rho)\rho_x$$

Multiplying eqn. (34) by  $C'(\rho) = C_\rho = C_\rho \cdot t_\rho$ , we get,

$$C_t + CC_x = \nu C'(\rho)\rho_{xx} = \nu \{C_{xx} - C''(\rho)\rho_x^2\} \quad (35)$$

If we take  $Q(\rho)$  as a quadratic function in  $\rho$ , then  $C(\rho)$  is linear in  $\rho$  and therefore  $C''(\rho) = 0$ . Therefore, equation (35) becomes

$$C_t + CC_x = \nu C_{xx} \quad (36)$$

For a simple model of turbulence, we have replaced  $C$  by the fluid velocity field  $u(x, t)$  to get

$$u_t + uu_x = \nu u_{xx} \quad (37)$$

where  $\nu$  is the kinematic viscosity.

Equation (37) is known as **Burgers equation**.

### Value Additions:

Burgers equation represents the balance between diffusion, non-linearity and time evolution. In fluid dynamics, for diffusive waves, the simplest nonlinear model equation is the Burgers equation.

**Example 5:** In fluid dynamics, derive the *continuity equation*

## Mathematical Models

$$\rho_t + \text{div}(\rho \vec{u}) = 0,$$

and *Euler's equation of motion*

$$\rho \left[ \vec{u}_t + (\vec{u} \cdot \text{grad}) \vec{u} \right] + \text{grad } p = 0,$$

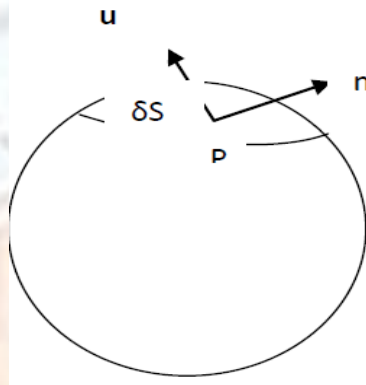


Fig 6: Flow from an incremental volume

**Solution:** Let  $S$  be closed surface drawn in the fluid. Let  $S$  contains volume  $V$  of the fluid. Let  $P(x, y, z)$  be any point of the fluid within  $S$ . Let  $\rho(x, y, z, t)$  be the fluid density at  $P$  at time  $t$ . Let  $\delta S$  denote elements of the surface  $S$  enclosing  $P$ . Let  $\hat{n}$  be the unit outward normal drawn at  $P$  and  $\vec{u}(x, y, z, t)$  be the fluid velocity at  $P$ . Then the normal component of  $\vec{u}$  measured outward from  $P$  is  $\vec{u} \cdot \hat{n}$ .

Mass of the fluid flowing out  $\delta S$  per unit time is equal to

$$\rho \cdot (\vec{u} \cdot \hat{n}) \delta S = \rho \vec{u} \cdot (\hat{n} \cdot \delta S) = \rho \vec{u} \cdot d\vec{S}$$

$\therefore$  Mass of the fluid flowing out of  $S$  per unit time is given by

$$\int_S \rho \vec{u} \cdot d\vec{S} = \int_V \nabla \cdot (\rho \vec{u}) dV$$

Also, mass of the fluid contained within the volume  $V$  at time  $t$  is  $\int_V \rho dV$

## Mathematical Models

∴ Rate of decrease of mass of the fluid within V is  $-\frac{\partial}{\partial t} \int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV$

By conservation law,

$$\frac{d}{dt} \int_V \rho dV = - \int_S \rho (\vec{u} \cdot \vec{n}) dS = - \int_S \vec{n} \rho \vec{u} dS = - \int_V \nabla (\rho \vec{u}) dV$$

(By divergence theorem)

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t} dV = - \int_V \nabla (\rho \vec{u}) dV,$$

$$\Rightarrow \int_V \left[ \frac{\partial \rho}{\partial t} + \nabla (\rho \vec{u}) \right] dV = 0$$

Since the integrand is continuous, therefore

$$\frac{\partial \rho}{\partial t} + \nabla (\rho \vec{u}) = 0$$

which is the equation of continuity.

**Euler's equation of motion:** Let a closed surface S, enclosing a volume V be moving with the fluid so that S contains the same number of fluid particles at any time t.

Consider a point P inside S. Let  $\rho$  is the fluid density,  $\vec{u}$  is the fluid velocity and  $dV$  be the elementary volume enclosing P.

Since the mass  $\rho dV$  remains unchanged throughout the motion, therefore

$$\frac{d}{dt} (\rho dV) = 0$$

Momentum = mass x velocity.

$$\therefore \vec{M} = \int_V \vec{u} \rho dV \Rightarrow \frac{d\vec{M}}{dt} = \int_V \vec{u} \rho dV = \int_V \frac{d\vec{u}}{dt} \rho dV$$

Let  $\hat{n}$  be the outward unit normal vector on the surface element  $dS$ . Suppose  $\vec{F}$  is the external force per unit mass acting on the fluid, and  $p$  be the pressure at any point on the element  $dS$ .

$\therefore$  Total surface force is

$$\int_V \vec{F} \cdot \rho dV + \int_S p(-\vec{n}) dS = \int_V \vec{F} \rho dV - \int_V \nabla p dV = \int_V (\vec{F} \rho - \nabla p) dV$$

By Newton's second law of motion,

Rate of change of momentum = Total applied force

$$\therefore \int_V \frac{d\vec{u}}{dt} \rho dV = \int_V (\vec{F} \rho - \nabla p) dV \Rightarrow \int_V \left( \frac{d\vec{u}}{dt} \rho - \vec{F} \rho + \nabla p \right) dV = 0$$

If the integrand is constant. Then,

$$\frac{d\vec{u}}{dt} \rho - \vec{F} \rho + \nabla p = 0$$

If no external force is acting then  $\vec{F} = 0$ .

$$\therefore \frac{d\vec{u}}{dt} \rho + \nabla p = 0$$

Now,  $\frac{d\vec{u}}{dt} = \left( \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right) \vec{u}$ , so by the above equation, we have

$$\rho \left[ \vec{u}_t + (\vec{u} \cdot \text{grad}) \vec{u} \right] + \text{grad } p = 0$$

### Summary:

We now end this chapter by giving a summary of it. In this chapter we have covered the following

- (1) The wave equation with reference to vibrations of stretched flexible string for one-dimensional case has been derived.

## Mathematical Models

- (2) The two dimensional wave equation that describes the deflection of a stretched vibrating membrane for any position and any time has been derived.
- (3) One - dimensional heat equation has been derived and our discussion of heat transfer has included heat conduction only.
- (4) The Laplace equation has been derived by choosing force of attraction of particles.
- (5) We have stated conservation law and derived the Burgers equation.

### Exercise:

1. While arriving at the equation describing the motion of the string, we assumed the weight to be negligible. Determine the describing equation including the weight of the string in the derivation.
2. Derive the one - dimensional heat equation

$$u_t = k u_{xx},$$

where  $k$  is a constant and also show that when the heat lost by radioactive exponential decay of the material in the bar is also considered then the one-dimensional heat equation becomes

$$u_t = k u_{xx} + h e^{-ax},$$

where  $h$  and  $a$  are constant.

3. Write the one - dimensional heat equation that could be used to determine the temperature in a flat circular disk with the flat surface insulated.
4. Express Laplace's equation using spherical coordinates.
5. Show that the Euler equation for an inviscid incompressible fluid flow under the body force,  $F = -\nabla\phi$ , are

$$u_t = u \cdot \nabla u = -\nabla\phi - \frac{1}{\rho} \nabla p, \quad \text{div} u = 0.$$

6. Verify that the vorticity  $\omega = \nabla \times u$  satisfies the vorticity equation



## Mathematical Models

$$\frac{D\omega}{Dt} = \frac{\partial\omega}{\partial t} + u \cdot \nabla\omega = \nabla u.$$

Give the interpretation of this vorticity equation. Also show that

$$\frac{D\omega}{Dt} = 0, \text{ in two dimensions.}$$

### Glossary:

**B** Burgers Equation: For a simple model of turbulence, second order partial differential equation denoted by  $u_t + uu_x = \nu u_{xx}$  where  $u(x, t)$  denote the fluid velocity and  $\nu$  is the kinematic viscosity is known as Burgers equation.

**G** Gravitational Potential: The Newton's gravitational law states that the attraction force between the particles is directly proportional to the product of  $m, M$  and  $1/r^2$ . So we have,

$$F \propto \frac{mM}{r^2} \text{ Consequently, } F = G \frac{mM}{r^2}$$

where  $G$  is the gravitational constant.

**L** Laplacian Operator: The operator  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$  is called Laplacian Operator. It is widely used in many physical phenomenon.

**O** One-dimensional Heat Equation: A second order partial differential equation  $\frac{\partial T}{\partial t} = K \nabla^2 T$  where  $K = \frac{k}{\sigma \delta}$ , is called one-dimensional heat equation.

**O** One-dimensional Wave Equation: A second order partial differential equation  $u_{tt} = c^2 u_{xx}$  where  $c^2 = \frac{T}{\rho}$ , is called the one-dimensional wave equation and  $c$  is called the wave speed.

T Two-dimensional Wave Equation: A second order partial differential equation  $u_{tt} = c^2 (u_{xx} + u_{yy})$  where  $c^2 = \frac{T}{\rho}$ , is called the two-dimensional wave equation.

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