Lesson: Numerical Solutions of Ordinary Differential Equations

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1. Learning Outcomes:

After studying this chapter, you should be able to

- identify the initial value problem for the first order ordinary differential equations;
- obtain the solution or the initial value problems by using Euler's method;
- obtain the solution of IVPs using Runge-Kutta methods of second and fourth order;
- extrapolate the approximate value of the solutions obtained by the Runge-Kutta methods of second and fourth order;

2. Introduction:

Solving differential equations, both ordinary and partial, is one of the most useful and important application of numerical analysis. There are mainly two common problems we face in finding the numerical solution of a differential equation. The first one is: When one finds a numerical solution is sufficiently close to the exact solution? The second problem is the instability of numerical solution. The actual solution to the problem of interest is stable, but the errors incurred in the numerical solution are magnified in such a way that the numerical solution is obviously incompatible with the actual solution. A numerical method that gives accurate results and is stable with the least amount of computation time often requires that it be started with a somewhat less accurate method and then continued with a more accurate technique. There are many starting techniques and methods that are used to continue a solution. In this chapter we shall introduce two such methods namely, Euler's method and Runge-Kutta Method of second order and fourth order to obtain numerical solution of ordinary differential equations (ODEs). Initially we shall focus on solving the first order equations, since, every nth–order equation is equivalent to a system of n first–order equations. To begin with, we shall recall few basic concepts from the theory of differential equations which we shall be referring quite often.

3. Basic Concepts:

In this section, we shall start with few definitions from differential equations and define some concepts which are involved in the numerical solution of differential equations.

Definition: An equation relating an unknown function (dependent variables) to its various derivatives with respect to known functions (independent variables) is called a **differential equation,** thus

$$\frac{d^2 f}{dx^2} + 2\,x\,f = e^x \qquad (1)$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (2)$$

are examples of differential equations.

Value Additions:
Differential equations involving derivatives w.r.t. a single independent variable are called **ordinary differential equations** (ODEs) whereas, A **partial differential equations** (PDEs) contains partial derivatives w.r.t. more than one independent variable. Eqn. **(1)** is an ODE, while Eqn. **(2)** is a PDE.

Definition: The **order** of a differential equation is the order of the highest derivative occurring in the equation and its **degree** is the highest exponent of the highest order derivative after the equation has been rationalised. The order and degree of the equation

$$\left(\frac{d^3u}{dx^3}\right)^4+2x\frac{d^2u}{dx^2}-\sin u=e^x \qquad (3)$$

is 3 and 4 respectively.

The **general solution** of an nth order linear equation is a family of solutions containing n arbitrary constants. In order to determine these arbitrary constants, n conditions are required. If these conditions are given at one point, then these conditions are known as **initial conditions** and the differential equation together with the initial conditions is called an **initial value problem** (IVP).

If the n conditions are prescribed at more than one point then these conditions are known as **boundary conditions.** The differential equation together with the boundary condition is known as a **boundary value problem** (BVP).

We will need to study numerical methods for the solution of the first order IVP

$$y' = f(y,t),\ y(t_0) = y_0 \qquad ...(4)$$

Starting with the initial values, the solution are hence constructed step by step through a series of equal intervals in the independent variables so that as soon as the solution has been carried to x = x_i; the next step will be to evaluate the change in the solution through the interval Δx = h of x_i to x_{i+1}.

Let us take the interval $[t_0, b]$ over which the solution of the IVP (4) is required. Sub-dividing the interval $[t_0, b]$ into n sub-intervals using a step size

$$h = \left[\frac{t_n - t_0}{n}\right], where\, t_n = b$$

We can then write $t_k = t_0 + kh,\ k = 0,1,.....n$. A numerical method for the solution of the IVP (4), will produce approximate values y_k, at the grid points t_k.

Let us now discuss how to construct numerical methods and related basic concepts with reference to a simple ODE

$$\frac{dy}{dt} = \lambda y,\ y(t_0) = y_0, t \in [a, b] \qquad (5)$$

Let us define the grid point by

$t_k = t_0 + kh,\ k = 0,1,.....n$, where $t_0 = a\, and\, t_0 + nh = b$

Separating the variables and integrating, we find that the exact solution of Eqn. (5) is

$$y(t) = y(t_0)e^{\lambda(t-t_0)} \qquad (6)$$

To get a relationship that connects two successive solution values, we put $t = t_n\ and\ t = t_{n+1}$ in Eqn. (6). Thus we get,

$$y(t_n) = y(t_0)e^{\lambda(t_n - t_0)}$$
and
$$y(t_{n+1}) = y(t_0)e^{\lambda(t_{n+1} - t_0)}$$

Dividing, we get,

$$\frac{y(t_{n+1})}{y(t_n)} = \frac{e^{\lambda t_{n+1}}}{e^{\lambda t_n}} = e^{\lambda(t_{n+1} - t_n)}$$

Hence we have,

$$y(t_{n+1}) = e^{\lambda h}\, y(t_n) \qquad (7)$$

Eqn. (7) gives the required relation between $y(t_n)$ *and* $y(t_{n+1})$.

Putting n = 0, 1, 2, ... we can find $y(t_1), y(t_2), y(t_3)$..... from the given value $y(t_0)$.

We can get a numerical method by approximating $e^{\lambda h}$ in Eqn. (7). We may use the following polynomial approximations.

$$e^{\lambda h} = 1 + \lambda h + o\left(|\lambda h|^2\right) \qquad (8)$$

$$e^{\lambda h} = 1 + \lambda h + \frac{\lambda^2 h^2}{2} + o\left(|\lambda h|^3\right) \qquad (9)$$

$$e^{\lambda h} = 1 + \lambda h + + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + o\left(|\lambda h|^4\right) \qquad (10)$$

and so on.

Now retain (p+l) terms in the expansion of $e^{\lambda h}$ and denote the approximation $e^{\lambda h}$ by $E(\lambda h)$. The numerical method for obtaining the approximate values t_n, of $y(t_n)$ can then be written as

$$y_{n+1} = E(\lambda h)\, y_n, n = 0, 1, 2, 3, ... \qquad (11)$$

4. Euler's Method:

Let us consider the first-order equation which can put in the form

$$\dot{y} = f(y, t) \qquad (12)$$

where $\dot{y} = \frac{dy}{dt}$. If y_i and $\dot{y}_i$ at t_i are known, then Eq. (12) can be used to give y_{i+1} and $\dot{y}_{i+1}$ at t_{i+1}.

Euler's method results from approximating the derivative

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \qquad (13)$$

By the difference equation

$$\Delta y \cong \dot{y}\, \Delta t \qquad (14)$$

Or, in the difference notation

$$y_{i+1} = h\, \dot{y}_i \qquad (15)$$

where $h = \Delta t = t_{i+1} - t_i$, is the step size.

Example 1: Use Euler's method to Solve the IVP $\frac{dy}{dt} = 1 - 2\,y\,t,\ y(0.2) = 0.1948.\ Find\ y(0.2)$ with $h = 0.2$.

Solutions: In Euler's method we must have the first derivative at each point; it is given by $\dot{y}_i = 1 - 2 y_i t_i$

The solution is approximated at each point by

$$y_{i+1} = y_i + h\,\dot{y}_i = y_i + h\left(1 - 2\,y_i\,t_i\right)$$

$$y(0.4) = 0.1948 + (0.2)\ (1 - 2 \text{ x } 0.2 \text{ x } 0.1948)$$

$$= 0.379216.$$

Example 2: Solve the IVP $y' = t + y,\ y(0) = 1,$ by Euler's method using $h = 0.1.$ Find the exact error, if the exact value is $y(1) = 3.436564.$

Solution: Euler's method is given by $y_{i+1} = y_i + h\,\dot{y}_i$

For our problem, we have $y_{i+1} = y_i + h\left(t_i + y_i\right) = (1 + h)\,y_i + h\,t_i$

Starting the iteration with $h = 0.1,\ y(0) = 1$, we get,

$$y_1 = (1 + 0.1) \times 1 + (0.1)\ (0) = 1.1$$

$$y_2 = (1.1)(1.1) + (0.1)(0.1) = 1.22,$$

$$y_3 = (1.1)(1.22) + (0.1)(0.2) = 1.362$$

$$y_4 = (1.1)(1.362) + (0.1)(0.3) = 1.5282,$$

$$y_5 = (1.1)(1.5282) + (0.1)(0.4) = 1.72102,$$

$$y_6 = (1.1)(1.72102) + (0.1)(0.5) = 1.943122,$$

$$y_7 = (1.1)(1.943122) + (0.1)(0.6) = 2.197434,$$

$$y_8 = (1.1)(2.197434) + (0.1)(0.7) = 2.487178,$$

$$y_9 = (1.1)(2.487178) + (0.1)(0.8) = 2.815895$$

$$y_{10} = (1.1)(2.815895) + (0.1)(0.9) = 3.187485 = y(1)$$

$$actual\ error = y(1) - y_{10} = 3.436564 - 3.187485 = 0.2491.$$

Example 3: Solve the IVP $3\frac{dy}{dt} + 5y^2 = \sin t, y(0.3) = 5$ by Euler's method using $h = 0.3.$ Find the value of $y(0.9)$.

Solution: First rewrite the differential equation in the proper form.

$$\frac{dy}{dt} = f(y,t) = \frac{1}{3}\left(\sin t - 5y^2\right), y(0.3) = 5$$

Euler's method is given by $y_{i+1} = y_i + h\, f(y_i, t_i)$

where

$$h = 0.3,\ for\ i = 0, t_0 = 0.3, y_0 = 5,$$

$$y_1 = y_0 + h\, f(y_0, t_0) = 5 + 0.3 \times \frac{1}{3}\left(\sin(0.3) - 5(5)^2\right) = -7.4704$$

Y_1 is the approximate value of *y* at

$$t = t_1 = t_0 + h = 0.3 + 0.3 = 0.6,\ for\ i = 1, t_1 = 0.6, y_1 = -7.4704,$$

$$y_2 = y_1 + h\, f(y_1, t_1) = -7.4704 + 0.3 \times \frac{1}{3}\left(\sin(0.6) - 5(-7.4704)^2\right) = -35.318$$

Y_2 is the approximate value of *y* at

$$t = t_2 = t_2 + h = 0.6 + 0.3 = 0.9,$$

$$y_2 = y(0.9) = -35.318$$

Example 4: Solve the IVP $\frac{dy}{dt} = \frac{4t}{y}, y(0) = 1$ by Euler's method using $h = 0.5.$

Find the value of $y(1).$

Solution: First rewrite the differential equation in the proper form.

$$\frac{dy}{dt} = f(y,t) = \frac{4t}{y}, y(0) = 1$$

Euler's method is given by $y_{i+1} = y_i + h\, f(y_i, t_i)$

where

$$h=0.5,\ for\ i=0, t_0=0, y_0=1,$$

$$y_1=y_0+h\ f\left(y_0,t_0\right)=1+0.5\times\frac{4\times 0}{1}=0$$

Y_1 is the approximate value of y at

$$t=t_1=t_0+h=0+0.5=0.5,\ for\ i=1, t_1=0.5, y_1=1,$$

$$y_2=y_1+h\ f\left(y_1,t_1\right)=1+0.5\times\frac{4\times 0.5}{1}=2$$

Y_2 is the approximate value of y at

$$t=t_2=t_2+h=0.5+0.5=1,$$
$$y_2=y\left(1\right)=2$$

Value Addition:
Euler’s method is simpler to use since we do not have to compute higher derivatives at each point. It could also be used to solve higher order equations.

5. Runge-Kutta Methods:

In order to produce accurate results using Taylor’s method, derivatives of higher order must be evaluated. This may be difficult, or the higher-order derivatives may be inaccurate. Methods that require only the first-order derivative and give results with the same order of truncation error as Taylor’s method maintain the higher-order derivatives are called the Runge-Kutta method. Estimates of the derivative must be made at points within each interval $t_i \le t \le t_{i+1}$. The prescribed first-order equation is used to provide the derivative at the interior points. The Runge-Kutta method of second-order will be developed and the Runge-Kutta method of fourth-order will simply be presented with no development.

Value Addition:
Since Euler's method is of first, it requires h to be very small to attain the desired accuracy. Hence, very often, the number of steps to be carried

out becomes very large. In such cases, we need higher order methods like Runge-Kutta to obtain the required accuracy in a limited number of steps.

5.1. Runge-Kutta Method of second order

Let us consider the first-order equation $\dot{y}=f(y,t)$. All Runge-Kutta methods utilize the approximation

$$y_{i+1}=h\phi_i\,, \qquad (16)$$

where ϕ_i is an approximation to the slope in the interval $t_i \le t \le t_{i+1}$.

Certainly, if we used $\phi_i = f_i$, the approximation for y_{i+1} would be too large for the curve in Figure 1; and, if we used $\phi_i = f_{i+1}$, the approximation would be too small. Hence, the correct ϕ_i needed to give the exact y_{i+1} lies in the interval $f_i \le \phi \le f_{i+1}$.

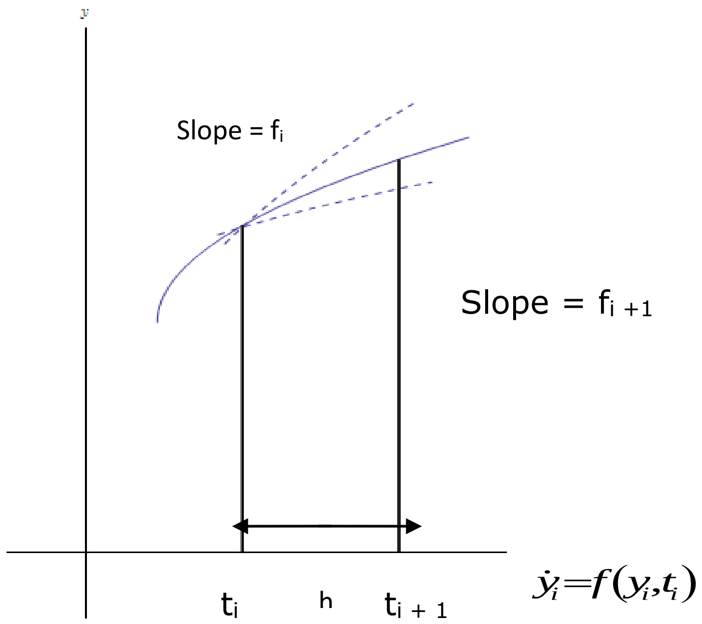

Figure 1. Curve showing approxi y_i ions to y_{i+1} sing slopes f_i and f_{i+1}

The trick is to find a technique that will give a good approximation to the correct slope ϕ_i. Let us assume that

$$\phi_i = a\,\xi_i + b\,\eta_i \tag{17}$$

$$\text{where } \xi_i = f\left(y_i, t_i\right) \tag{18}$$

$$\eta_i = f\left(y_i + q\,h\,\xi_i, t_i + p\,h\right) \tag{19}$$

The quantities a, b, p, and q are constant to be established later.

A good approximation for η_i is found by expanding in a Taylor series, neglecting higher-order terms:

$$\begin{aligned}\eta_i &= f\left(y_i, t_i\right) + \frac{\partial f}{\partial y}\left(y_i, t_i\right)\Delta y + \frac{\partial f}{\partial t}\left(y_i, t_i\right)\Delta t + o(\mathrm{h}^2)\\ &= f_i + q\,h\,f_i\,\frac{\partial f}{\partial y}\left(y_i, t_i\right) + p\,h\,\frac{\partial f}{\partial t}\left(y_i, t_i\right)\Delta t + o(\mathrm{h}^2)\end{aligned} \tag{20}$$

Where we have used $\Delta y = qhf_i$ *and* $\Delta t = ph,$ as required by eqn. (19). Eqn. (16) then becomes, using $\xi_i = f_i$,

$$\begin{aligned}y_{i+1} &= y_i + h\phi_i = y_i + h\left(a\,\xi_i + b\,\eta_i\right)\\ &= y_i + h\left(a\,f_i + b\,f_i\right) + h^2\left[b\,q\,f_i\,\frac{\partial f}{\partial y}\left(y_i, t_i\right) + b\,p\,f_i\,\frac{\partial f}{\partial t}\left(y_i, t_i\right)\right] + o(h^3),\end{aligned} \tag{21}$$

where we have substituted for ξ_i and η_i from eqn. (18) and (20), respectively. Expand y_i in Taylor series with second order, so that

$$\begin{aligned}y_{i+1} &= y_i + h\,\dot{y}_i + \frac{h^2}{2}\,\ddot{y}_i\\ &= y_i + h\,f\left(y_i, t_i\right) + \frac{h^2}{2}\,\dot{f}\left(y_i, t_i\right)\end{aligned} \tag{22}$$

Now, using the chain rule,

$$\dot{f} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial t} = \dot{y}\,\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t} = f\,\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t} \tag{23}$$

Thus, we have

$$y_{i+1} = y_i + h\, f\left(y_i, t_i\right) + \frac{h^2}{2}\left[f_i \frac{\partial f}{\partial y}\left(y_i, t_i\right) + \frac{\partial f}{\partial t}\left(y_i, t_i\right) \right] \qquad (24)$$

Comparing this with eqn. (21), we find that (equating terms in like powers of h)

$$a + b = 1, \quad b\,q = ½, \quad b\,p = ½ \qquad (25)$$

These three equations contain four unknowns; hence one of them is arbitrary. It is compulsory to choose b = ½ or b = 1. For b = ½, we have a = ½, q = 1 and p = 1. Then our approximation for y_{i+1} from eqn. (21) becomes

$$\begin{aligned} y_{i+1} &= y_i + h\left(a\,\xi_i + b\,\eta_i\right) \\ &= y_i + \frac{h}{2}\left[f\left(y_i, t_i\right) + f\left(y_i + h\, f_i, t_i + h\right) \right] \end{aligned} \qquad (26)$$

For b = 1, we have a = 0, q = ½, and p = ½, these results

$$y_{i+1} = y_i + h\,\eta_i = y_i + h\, f\left(y_i + \frac{h}{2} f_i, t_i + \frac{h}{2} \right) \qquad (27)$$

Knowing y_i, t_i and $\dot{y}_i = f_i$ we can now calculate y_{i+1} with the same accuracy obtained using Taylor's method that required us to know $\ddot{y}_i$.

Value Addition:
The different R-K methods are of second order are given by (1) **Optimal R-K method**: $y_{i+1} = y_i + \frac{1}{4}\left[\xi_i + 3\eta_i\right] \qquad (28)$ where $\xi_i = h\, f\left(y_i, t_i\right)$ $\eta_i = h\, f\left(y_i + \frac{2\xi_i}{3}, t_i + \frac{2h}{3} \right)$

(2) **Improved tangent method:**

$$y_{i+1} = y_i + \eta_i \qquad (29)$$

where

$$\xi_i = h\, f\left(y_i, t_i\right)$$

$$\eta_i = h\, f\left(y_i + \frac{\xi_i}{2}, t_i + \frac{h}{2}\right)$$

This method is also known as **modified Euler's method.**

(3) **Heun's method:**

$$y_{i+1} = y_i + \frac{1}{2}\left(\xi_i + \eta_i\right) \qquad (30)$$

where

$$\xi_i = h\, f\left(y_i, t_i\right)$$

$$\eta_i = h\, f\left(y_i + \xi_i, t_i + h\right)$$

This method is also known as the **Euler-Cauchy method.**

Example 5: Solve the IVP $y' = 1 + y^2, y(0) = 0$ and find $y(0.4)$ *with* $h = 0.2$ using the following **R-K** methods of second order

a) Optimal R-K method

b) Improved tangent method

c) Heun's method

Compare the results with the exact solution $y(t) = \tan t$, and find the errors.

Solutions: a) Optimal **Runge-Kutta,** method:

$$\xi_0 = h\, f\left(y_0, t_0\right) = h\left(1 + {y_0}^2\right) = 0.2\left(1 + 0\right) = 0.2$$

$$\eta_0 = h\, f\left(y_0 + \frac{2\xi_0}{3}, t_0 + \frac{2h}{3}\right) = h\left[1 + \left(y_0 + \frac{2\xi_0}{3}\right)^2\right] = 0.2\left[1 + \left(0 + \frac{2 \times 0.2}{3}\right)^2\right]$$

$$= 0.2035556$$

Thus,

$$y_1 = y(0.2) = y_0 + \frac{1}{4}[\xi_0 + 3\eta_0] = 0 + \frac{1}{4}[0.2 + 3\times 0.2035556] = 0.2026667$$

$To\, calculate\, y_2, we\, need,$

$$\xi_1 = h\, f\,(y_1, t_1) = h\left(1 + y_1^{\,2}\right) = 0.2\left[1 + (0.2026667)^2\right] = 0.2082148$$

$$\eta_1 = h\, f\left(y_1 + \frac{2\xi_1}{3}, t_1 + \frac{2h}{3}\right)$$

$$= h\left[1 + \left(y_1 + \frac{2\xi_1}{3}\right)^2\right]$$

$$= 0.2\left[1 + \left(0.2026667 + \frac{2\times 0.2082148}{3}\right)^2\right]$$

$$= 0.223321245$$

Thus,

$$y_2 = y(0.4)$$

$$= y_1 + \frac{1}{4}[\xi_1 + 3\eta_1] = 0.2026667 + \frac{1}{4}[0.2082148 + 3\times 0.223321245]$$

$$= 0.422211334$$

b) Improved tangent method is

$$\xi_0 = h\, f\,(y_0, t_0) = h\left(1 + y_0^{\,2}\right) = 0.2(1+0) = 0.2,$$

$$\eta_0 = h\, f\left(y_0 + \frac{\xi_0}{2}, t_0 + \frac{h}{2}\right) = h\left[1 + \left(y_0 + \frac{\xi_0}{2}\right)^2\right] = 0.2\left[1 + \left(0 + \frac{0.2}{2}\right)^2\right] = 0.202$$

$$y_1 = y(0.2) = y_0 + \eta_0 = 0 + 0.202 = 0.202$$

$$\xi_1 = h\, f\,(y_1, t_1) = h\left(1 + y_1^{\,2}\right) = 0.2\left[1 + (0.202)^2\right] = 0.2081608,$$

$$\eta_1 = h\, f\left(y_1 + \frac{\xi_1}{2}, t_1 + \frac{h}{2}\right) = h\left[1 + \left(y_1 + \frac{\xi_1}{2}\right)^2\right] = 0.2\left[1 + \left(0.202 + \frac{0.2081608}{2}\right)^2\right]$$

$$= 0.21873704$$

$$y_2 = y(0.4) = y_1 + \eta_1 = 0.202 + 0.21873704 = 0.42073704$$

c) Heun's method :

$$\xi_0 = h\, f\,(y_0, t_0) = h\left(1 + y_0^{\,2}\right) = 0.2(1+0) = 0.2$$

$$\eta_0 = h\, f\,(y_0 + \xi_0, t_0 + h) = h\left[1 + (y_0 + \xi_0)^2\right] = 0.2\left[1 + (0 + 0.2)^2\right] = 0.208$$

$$y_1 = y(0.2) = y_0 + \frac{1}{2}(\xi_0 + \eta_0) = 0 + \frac{1}{2}(0.2 + 0.208) = 0.204$$

$$\xi_1 = h\, f\left(y_1, t_1\right) = h\left(1 + y_1^{\,2}\right) = 0.2\left[1 + \left(0.204\right)^2\right] = 0.2083232$$

$$\eta_1 = h\, f\left(y_1 + \xi_1\ , t_1 + h\right) = h\left[1 + \left(y_1 + \xi_1\right)^2\right] = 0.2\left[1 + \left(0.204 + 0.2083232\right)^2\right]$$
$$= 0.2340020843$$

$$y_2 = y\left(0.4\right) = y_1 + \frac{1}{2}\left(\xi_1 + \eta_1\right) = 0.204 + \frac{1}{2}\left(0.2083232 + 0.2340020843\right)$$
$$= 0.4251626422$$

Now the exact solution is y(t) = tan t

Exact y(0.4) = 0.422793219

Error in Optimal R-K method = 0.582×10^{-3}

Error in Improved tangent method= 0.205×10^{-2}

Error in Heun's method = 0.236×10^{-2}

Example 6: Solve the IVP $y' = t^2 + y^2, y(0) = 1$ and find $y\left(0.2\right)$ *with* $h = 0.1$ using the following **R-K** methods of second order

a) Optimal R-K method

b) Improved tangent method

c) Heun's method

Solution: a) Optimal **Runge-Kutta,** method:

$$\xi_0 = h\, f\left(y_0, t_0\right) = h\left(t_0^{\,2} + y_0^{\,2}\right) = 0.1\left(0 + 1\right) = 0.1$$

$$\eta_0 = h\, f\left(y_0 + \frac{2\xi_0}{3}\ , t_0 + \frac{2h}{3}\right) = h\left[\left(t_0 + \frac{2h}{3}\right)^2 + \left(y_0 + \frac{2\xi_0}{3}\right)^2\right]$$

$$= 0.1\left[\left(0 + \frac{2\times 0.1}{3}\right)^2 + \left(1 + \frac{2\times 0.1}{3}\right)^2\right] = 0.11422222$$

Thus,

$$y_1 = y\left(0.1\right) = y_0 + \frac{1}{4}\left[\xi_0 + 3\eta_0\ \right] = 1 + \frac{1}{4}\left[\ 0.1 + 3\times 0.11422222\ \right] = 1.11066667$$

To calculate y_2, we need,

$$\xi_1 = h\, f\left(y_1, t_1\right) = h\left(t_1^{\,2} + y_1^{\,2}\right) = 0.1\left[(0.1)^2 + (1.11066667)^2\right] = 0.12435805$$

$$\eta_1 = h\,f\left(y_1 + \frac{2\xi_1}{3}, t_1 + \frac{2h}{3}\right) = h\left[\left(t_1 + \frac{2h}{3}\right)^2 + \left(y_1 + \frac{2\xi_1}{3}\right)^2\right]$$

$$= 0.1\left[\left(0.1 + \frac{2\times 0.1}{3}\right)^2 + \left(1.11066667 + \frac{2\times 0.12435805}{3}\right)^2\right] = 0.1452392$$

Thus,

$$y_2 = y(0.2) = y_1 + \frac{1}{4}[\xi_1 + 3\eta_1] = 1.11066667 + \frac{1}{4}[\,0.12435805 + 3\times 0.1452392\,]$$
$$= 1.25068558$$

b) Improved tangent method is

$$\xi_0 = h\,f(y_0, t_0) = h\left(t_0^2 + y_0^2\right) = 0.1(1+0) = 0.1,$$

$$\eta_0 = h\,f\left(y_0 + \frac{\xi_0}{2}, t_0 + \frac{h}{2}\right) = h\left[\left(t_0 + \frac{h}{2}\right)^2 + \left(y_0 + \frac{\xi_0}{2}\right)^2\right]$$

$$= 0.1\left[0.0025 + \left(1 + \frac{0.1}{2}\right)^2\right] = 0.1105$$

$$y_1 = y(0.1) = y_0 + \eta_0 = 1 + 0.1105 = 1.1105$$

$$\xi_1 = h\,f(y_1, t_1) = h\left(t_1^2 + y_1^2\right) = 0.1\left[0.01 + (1.1105)^2\right] = 0.12432,$$

$$\eta_1 = h\,f\left(y_1 + \frac{\xi_1}{2}, t_1 + \frac{h}{2}\right) = h\left[\left(t_1 + \frac{h}{2}\right)^2 + \left(y_1 + \frac{\xi_1}{2}\right)^2\right]$$

$$= 0.1\left[\left(0.1 + \frac{0.1}{2}\right)^2 + \left(1.1105 + \frac{0.12432}{2}\right)^2\right] = 0.13976$$

$$y_2 = y(0.2) = y_1 + \eta_1 = 1.1105 + 0.13976 = 1.25026$$

c) Heun's method :

$$\xi_0 = h\,f(y_0, t_0) = h\left(t_0^2 + y_0^2\right) = 0.1(0+1) = 0.1$$

$$\eta_0 = h\,f(y_0 + \xi_0, t_0 + h) = h\left[(t_0 + h)^2 + (y_0 + \xi_0)^2\right] = 0.1[0.01 + 1.21] = 0.122$$

$$y_1 = y(0.1) = y_0 + \frac{1}{2}(\xi_0 + \eta_0) = 1 + \frac{1}{2}(0.1 + 0.122) = 1.111$$

$$\xi_1 = h\,f(y_1, t_1) = h\left(t_1^2 + y_1^2\right) = 0.1\left[0.01 + (1.111)^2\right] = 0.1244321$$

$$\eta_1 = h\,f(y_1 + \xi_1, t_1 + h) = h\left[(t_1 + h)^2 + (y_1 + \xi_1)^2\right]$$
$$= 0.1[0.04 + 1.52629247] = 0.15662925$$

$$y_2 = y(0.2) = y_1 + \frac{1}{2}(\xi_1 + \eta_1) = 1.111 + \frac{1}{2}(0.1244321 + 0.15662925) = 1.25153068$$

Example 7: Solve the IVP $y' = -t\,y^2, y(2) = 1$ and find $y(2.2)$ *with* $h = 0.1$ using the following **R-K** methods of second order

a) Optimal R-K method

b) Improved tangent method

c) Heun's method

Solutions: a) Optimal **Runge-Kutta,** method:

$$\xi_0 = h\,f(y_0, t_0) = h\left(-t_0\,y_0^2\right) = 0.1(-2) = -0.2$$

$$\eta_0 = h\,f\left(y_0 + \frac{2\xi_0}{3}, t_0 + \frac{2h}{3}\right) = h\left[-\left(t_0 + \frac{2h}{3}\right)\left(y_0 + \frac{2\xi_0}{3}\right)^2\right]$$

$$= 0.1\left[-\left(2 + \frac{0.2}{3}\right)\left(1 - \frac{2\times 0.2}{3}\right)^2\right] = -0.15522963$$

$$Thus,\ y_1 = y(2.1) = y_0 + \frac{1}{4}[\xi_0 + 3\eta_0] = 1 + \frac{1}{4}[-0.2 - 3\times 0.15522963] = 0.83357778$$

To calculate y_2, *we need,*

$$\xi_1 = h\,f(y_1, t_1) = h\left(-t_1\,y_1^2\right) = 0.1\left[-2.1(0.83357778)^2\right] = -0.1459189$$

$$\eta_1 = h\,f\left(y_1 + \frac{2\xi_1}{3}, t_1 + \frac{2h}{3}\right) = h\left[-\left(t_1 + \frac{2h}{3}\right)\left(y_1 + \frac{2\xi_1}{3}\right)^2\right]$$

$$= 0.1\left[-\left(2.1 + \frac{0.2}{3}\right)\left(0.83357778 - \frac{2\times 0.1459189}{3}\right)^2\right] = -0.11746269$$

$$Thus,\ y_2 = y(2.2) = y_1 + \frac{1}{4}[\xi_1 + 3\eta_1]$$

$$= 0.83357778 + \frac{1}{4}[-0.1459189 - 3\times 0.11746269] = 0.70900104$$

b) Improved tangent method is

$$\xi_0 = h\,f(y_0, t_0) = h\left(-t_0\,y_0^2\right) = 0.1(-2) = -0.2,$$

$$\eta_0 = h\,f\left(y_0 + \frac{\xi_0}{2}, t_0 + \frac{h}{2}\right) = h\left[-\left(t_0 + \frac{h}{2}\right)\left(y_0 + \frac{\xi_0}{2}\right)^2\right]$$

$$= 0.1\left[-\left(2 + \frac{0.1}{2}\right)\left(1 - \frac{0.2}{2}\right)^2\right] = -0.16605$$

$$y_1 = y(2.1) = y_0 + \eta_0 = 1 - 0.16605 = 0.83395$$

$$\xi_1 = h\, f\left(y_1, t_1\right) = h\left(-t_1\, y_1^{\,2}\right) = 0.1\left[-2.1(0.83395)^2\right] = -0.14604925,$$

$$\eta_1 = h\, f\left(y_1 + \frac{\xi_1}{2}\,, t_1 + \frac{h}{2}\right) = h\left[-\left(t_1 + \frac{h}{2}\right)\left(y_1 + \frac{\xi_1}{2}\right)^2\right]$$

$$= 0.1\left[-2.15\left(0.83395 - \frac{0.14604925}{2}\right)^2\right] = -0.1244866$$

$$y_2 = y\left(2.2\right) = y_1 + \eta_1 = 0.83395 - 0.1244866 = 0.7094634$$

c) Heun's method :

$$\xi_0 = h\, f\left(y_0, t_0\right) = h\left(-t_0\, y_0^{\,2}\right) = 0.1\left(2\right) = -0.2$$

$$\eta_0 = h\, f\left(y_0 + \xi_0\,, t_0 + h\right) = h\left[-\left(t_0 + h\right)\left(y_0 + \xi_0\right)^2\right] = 0.1\left[-2.1\left(1 - 0.2\right)^2\right] = -0.1344$$

$$y_1 = y\left(2.1\right) = y_0 + \frac{1}{2}\left(\xi_0 + \eta_0\right) = 1 + \frac{1}{2}\left(-0.2 - 0.1344\right) = 0.8328$$

$$\xi_1 = h\, f\left(y_1, t_1\right) = h\left(-t_1\, y_1^{\,2}\right) = 0.1\left[-2.1\left(0.8328\right)^2\right] = -0.14564673$$

$$\eta_1 = h\, f\left(y_1 + \xi_1\,, t_1 + h\right) = h\left[-\left(t_1 + h\right)\left(y_1 + \xi_1\right)^2\right]$$

$$= 0.1\left[-2.22\left(0.8328 - 0.14564673\right)^2\right] = -0.10482388$$

$$y_2 = y\left(2.2\right) = y_1 + \frac{1}{2}\left(\xi_1 + \eta_1\right) = 0.8328 + \frac{1}{2}\left(-0.14564673 - 0.10482388\right) = 0.70756469$$

Example 8: Solve the IVP $y' = 3t + \frac{y}{2}, y(0) = 1$ and find $y\left(0.2\right)$ *with* $h = 0.1$ using the following **R-K** methods of second order

a) Optimal R-K method

b) Improved tangent method

c) Heun's method

Solution: a) Optimal **Runge-Kutta,** method:

$$\xi_0 = h\, f\left(y_0, t_0\right) = h\left(3t_0 + \frac{y_0}{2}\right) = 0.1\left(0 + 0.5\right) = 0.05$$

$$\eta_0 = h\, f\left(y_0 + \frac{2\,\xi_0}{3}\,, t_0 + \frac{2\,h}{3}\right) = h\left[3\left(t_0 + \frac{2\,h}{3}\right) + \left(\frac{y_0}{2} + \frac{\xi_0}{3}\right)\right]$$

$$= 0.1\left[3\left(0 + \frac{2\times 0.1}{3}\right) + \left(0.5 + \frac{0.05}{3}\right)\right] = 0.07166667$$

Thus,

$$y_1 = y(0.1) = y_0 + \frac{1}{4}[\xi_0 + 3\eta_0] = 1 + \frac{1}{4}[0.05 + 3 \times 0.07166667] = 1.06625$$

$$To\, calculate\; y_2, we\, need,$$

$$\xi_1 = h\,f(y_1, t_1) = h\left(3t_1 + \frac{y_1}{2}\right) = 0.1[0.3 + 0.533125] = 0.0833125$$

$$\eta_1 = h\,f\left(y_1 + \frac{2\xi_1}{3}, t_1 + \frac{2h}{3}\right) = h\left[3\left(t_1 + \frac{2h}{3}\right) + \left(\frac{y_1}{2} + \frac{\xi_1}{3}\right)\right]$$

$$= 0.1\left[0.5 + \left(0.533125 + \frac{0.0833125}{3}\right)\right] = 0.10608958$$

Thus,

$$y_2 = y(0.2) = y_1 + \frac{1}{4}[\xi_1 + 3\eta_1] = 1.06625 + \frac{1}{4}[0.0833125 + 3 \times 0.10608958]$$

$$= 1.16664531$$

b) Improved tangent method is

$$\xi_0 = h\,f(y_0, t_0) = h\left(3t_0 + \frac{y_0}{2}\right) = 0.1(0 + 0.5) = 0.05$$

$$\eta_0 = h\,f\left(y_0 + \frac{\xi_0}{2}, t_0 + \frac{h}{2}\right) = h\left[3\left(t_0 + \frac{h}{2}\right) + \left(\frac{y_0}{2} + \frac{\xi_0}{4}\right)\right]$$

$$y_1 = y(0.1) = y_0 + \eta_0 = 1 + 0.05 = 1.05$$

$$\xi_1 = h\,f(y_1, t_1) = h\left(3t_1 + \frac{y_1}{2}\right) = 0.1[0.3 + 0.525] = 0.0825,$$

$$\eta_1 = h\,f\left(y_1 + \frac{\xi_1}{2}, t_1 + \frac{h}{2}\right) = h\left[3\left(t_1 + \frac{h}{2}\right) + \left(\frac{y_1}{2} + \frac{\xi_1}{4}\right)\right]$$

$$= 0.1\left[3 \times 0.15 + \left(0.525 + \frac{0.0825}{2}\right)\right] = 0.09956$$

$$y_2 = y(0.2) = y_1 + \eta_1 = 1.05 + 0.09956 = 1.14956$$

c) Heun's method :

$$\xi_0 = h\,f(y_0, t_0) = h\left(1 + y_0^2\right) = 0.2(1 + 0) = 0.2$$

$$\eta_0 = h\,f(y_0 + \xi_0, t_0 + h) = h\left[1 + (y_0 + \xi_0)^2\right] = 0.2\left[1 + (0 + 0.2)^2\right] = 0.208$$

$$y_1 = y(0.2) = y_0 + \frac{1}{2}(\xi_0 + \eta_0) = 0 + \frac{1}{2}(0.2 + 0.208) = 0.204$$

$$\xi_1 = h\,f(y_1, t_1) = h\left(1 + y_1^2\right) = 0.2\left[1 + (0.204)^2\right] = 0.2083232$$

$$\eta_1 = h\, f\left(y_1 + \xi_1 \,, t_1 + h\right) = h\left[1 + \left(y_1 + \xi_1\right)^2\right] = 0.2\left[1 + \left(0.204 + 0.2083232\right)^2\right]$$

$$= 0.2340020843$$

$$y_2 = y\left(0.4\right) = y_1 + \frac{1}{2}\left(\xi_1 + \eta_1\right) = 0.204 + \frac{1}{2}\left(0.2083232 + 0.2340020843\right)$$

$$= 0.4251626422$$

5.2. Runge-Kutta Method of Fourth order

The Runge-Kutta Method of Fourth order is perhaps the most widely used method for solving ODEs. One such method results in

$$y_{i+1} = y_i + \frac{h}{6}\left[\xi_i + \left(2 - \sqrt{2}\right)\eta_i + \left(2 + \sqrt{2}\right)\zeta_i + w_i\right] \qquad (31)$$

where

$$\begin{aligned}
\xi_i &= f\left(y_i, t_i\right) \\
\eta_i &= f\left(y_i + \frac{h}{2}\xi_i \,, t_i + \frac{h}{2}\right) \\
\zeta_i &= f\left[y_i + \frac{h}{\sqrt{2}}\left(\xi_i - \eta_i\right) - \frac{h}{2}\left(\xi_i - 2\eta_i\right), t_i + \frac{h}{2}\right] \qquad (32) \\
w_i &= f\left[y_i - \frac{h}{\sqrt{2}}\left(\eta_i - \zeta_i\right) + h\,\zeta_i, t_i + h\right].
\end{aligned}$$

This method is known as **Runge-Kutta-GILL method.**

Another method with order four Known as **Classical Runge-Kutta method,** is widely used method due to its simplicity and moderate order. We shall also be working out problems mostly by the classical R-K method unless specified otherwise. This method is given by

$$y_{i+1} = y_i + \frac{h}{6}\left[\xi_i + 2\eta_i + 2\zeta_i + w_i\right] \qquad (33)$$

where

$$\begin{aligned}
\xi_i &= f\left(y_i, t_i\right) \\
\eta_i &= f\left(y_i + \frac{h}{2}\xi_i, t_i + \frac{h}{2}\right) \\
\zeta_i &= f\left(y_i + \frac{h}{2}\eta_i, t_i + \frac{h}{2}\right) \qquad (34) \\
w_i &= f\left(y_i + h\zeta_i, t_i + h\right)
\end{aligned}$$

In all methods above no information is needed other than the initial condition. For example, y_i is approximated by using y_0, ξ_0, η_0, and so on. The quantities are found from the given equation with no differentiation required. These reasons, combined with the accuracy of Runge-Kutta methods, make them extremely popular.

Example 9: Use Classical Runge-Kutta method of fourth-order to solve the IVP $\dot{y} = 4 - 2yt, y(0) = 0.2 \text{ using } \mathrm{h} = 0.1.$ Carry out the solution for two time steps.

Solution: The first derivative is found from $\dot{y}_i = 4 - 2y_i t_i$. To find y_i we must know $y_0, \xi_0, \eta_0, \zeta_0$ and w_0. They are

$$\begin{aligned}
& y_0 = 0.2, \\
& \xi_0 = f\left(y_0, t_0\right) = 4 - 2y_0 t_0 = 4 - 2\times 0.2 \times 0 = 4, \\
& \eta_0 = f\left(y_0 + \frac{h}{2}\xi_0, t_0 + \frac{h}{2}\right) = 4 - 2\left(y_0 + \frac{h}{2}\xi_0\right)\left(t_0 + \frac{h}{2}\right) \\
& \quad = 4 - 2\left(0.2 + \frac{0.1}{2}\times 4\right)\left(\frac{0.1}{2}\right) = 3.96, \\
& \zeta_0 = f\left(y_0 + \frac{h}{2}\eta_0, t_0 + \frac{h}{2}\right) = 4 - 2\left(y_0 + \frac{h}{2}\eta_0\right)\left(t_0 + \frac{h}{2}\right) \\
& \quad = 4 - 2\left(0.2 + \frac{0.1}{2}\times 3.96\right)\left(\frac{0.1}{2}\right) = 3.96, \\
& w_0 = f\left(y_0 + h\zeta_0, t_0 + h\right) = 4 - 2\left(y_0 + h\zeta_0\right)\left(t_0 + h\right) \\
& \quad = 4 - 2\left(0.2 + 0.1\times 3.96\right)\left(0.1\right) = 3.88
\end{aligned}$$

Thus,

$$y_1 = y_0 + \frac{h}{6}[\xi_0 + 2\eta_0 + 2\zeta_0 + w_0]$$

$$= 0.2 + \frac{0.1}{6}(3.96 + 7.92 + 7.92 + 3.88) = 0.595$$

For next iteration, we start from y_1, and calculate the values of ξ_1, η_1, ζ_1 and w_1. They are

$$\xi_1 = f(y_1, t_1) = 4 - 2y_1 t_1 = 4 - 2\times 0.595\times 0.1 = 3.88,$$
$$\eta_1 = f\left(y_1 + \frac{h}{2}\xi_1, t_1 + \frac{h}{2}\right) = 4 - 2\left(y_1 + \frac{h}{2}\xi_1\right)\left(t_1 + \frac{h}{2}\right)$$
$$= 4 - 2\left(0.595 + \frac{0.1}{2}\times 3.88\right)\left(0.1 + \frac{0.1}{2}\right) = 3.76,$$
$$\zeta_1 = f\left(y_1 + \frac{h}{2}\eta_1, t_1 + \frac{h}{2}\right) = 4 - 2\left(y_1 + \frac{h}{2}\eta_1\right)\left(t_1 + \frac{h}{2}\right)$$
$$= 4 - 2\left(0.595 + \frac{0.1}{2}\times 3.76\right)\left(0.1 + \frac{0.1}{2}\right) = 3.77,$$
$$w_1 = f(y_1 + h\zeta_1, t_1 + h) = 4 - 2(y_1 + h\zeta_1)(t_1 + h)$$
$$= 4 - 2(0.595 + 0.1\times 3.77)(0.1 + 01) = 3.61$$

Thus,

$$y_2 = y_1 + \frac{h}{6}[\xi_1 + 2\eta_1 + 2\zeta_1 + w_1]$$
$$= 0.595 + \frac{0.1}{6}(3.88 + 7.52 + 7.54 + 3.61) = 0.971$$

Example 10: Use Classical Runge-Kutta method of fourth-order to solve the IVP $\dot{y} = y + t,\ y(0) = 1 \text{ using } \mathrm{h} = 0.1.$ Carry out the solution for two time steps.

Solution: The first derivative is found from $\dot{y}_i = y_i + t_i$. To find y_i we must know y_0, ξ_0, η_0, ζ_0 and w_0. They are

$$y_0 = 1,$$
$$\xi_0 = f(y_0, t_0) = y_0 + t_0 = 1 + 0 = 1,$$
$$\eta_0 = f\left(y_0 + \frac{h}{2}\xi_0, t_0 + \frac{h}{2}\right) = \left(y_0 + \frac{h}{2}\xi_0\right) + \left(t_0 + \frac{h}{2}\right) = \left(1 + \frac{0.1}{2}\times 1\right) + \left(\frac{0.1}{2}\right) = 1.1,$$

$$\zeta_0 = f\left(y_0 + \frac{h}{2}\eta_0, t_0 + \frac{h}{2}\right) = \left(y_0 + \frac{h}{2}\eta_0\right) + \left(t_0 + \frac{h}{2}\right) = \left(1 + \frac{0.1}{2}\times 1.1\right) + \left(\frac{0.1}{2}\right) = 1.105,$$

Thus,

$$y_1 = y_0 + \frac{h}{6}[\xi_0 + 2\eta_0 + 2\zeta_0 + w_0] = 1 + \frac{0.1}{6}(1 + 2.2 + 2.210 + 1.2105) = 1.11034$$

$$w_0 = f\left(y_0 + h\zeta_0, t_0 + h\right) = \left(y_0 + h\zeta_0\right) + \left(t_0 + h\right) = \left(1 + 0.1\times 1.105\right) + \left(0.1\right) = 1.2105$$

For next iteration, we start from y_1, and calculate the values of ξ_1, η_1, ζ_1 and w_1. They are

$$\xi_1 = f\left(y_1, t_1\right) = y_1 + t_1 = 1.11034 + 0.1 = 1.21034,$$
$$\eta_1 = f\left(y_1 + \frac{h}{2}\xi_1, t_1 + \frac{h}{2}\right) = \left(y_1 + \frac{h}{2}\xi_1\right) + \left(t_1 + \frac{h}{2}\right)$$
$$= \left(1.11034 + \frac{0.1}{2}\times 1.21034\right) + \left(0.1 + \frac{0.1}{2}\right) = 1.320857,$$
$$\zeta_1 = f\left(y_1 + \frac{h}{2}\eta_1, t_1 + \frac{h}{2}\right) = \left(y_1 + \frac{h}{2}\eta_1\right) + \left(t_1 + \frac{h}{2}\right)$$
$$= \left(1.11034 + \frac{0.1}{2}\times 1.320857\right) + \left(0.1 + \frac{0.1}{2}\right) = 1.32638285,$$
$$w_1 = f\left(y_1 + h\zeta_1, t_1 + h\right) = \left(y_1 + h\zeta_1\right) + \left(t_1 + h\right)$$
$$= \left(1.11034 + 0.1\times 1.32638285\right) + \left(0.1 + 01\right) = 1.44297829$$

Thus,

$$y_2 = y_1 + \frac{h}{6}[\xi_1 + 2\eta_1 + 2\zeta_1 + w_1]$$
$$= 1.11034 + \frac{0.1}{6}(1.21034 + 2.641714 + 2.6527657 + 1.44297829) = 1.2428033$$

Example 11: Use Classical Runge-Kutta method of fourth-order to solve the IVP $\dot{y} = 2y + 3e^t$, $y(0) = 0$ using $\mathrm{h} = 0.1$. Carry out the solution for two time steps.

Solution: The first derivative is found from $\dot{y}_i = 2y_i + 3e^{t_i}$. To find y_i we must know y_0, ξ_0, η_0, ζ_0 and w_0. They are

$$y_0 = 0,$$
$$\xi_0 = f(y_0, t_0) = 2y_0 + 3e^{t_0} = 2\times 0 + 3\times e^0 = 3,$$

$$\eta_0 = f\left(y_0 + \frac{h}{2}\xi_0, t_0 + \frac{h}{2}\right) = 2\left(y_0 + \frac{h}{2}\xi_0\right) + 3e^{\left(t_0+\frac{h}{2}\right)} = 2\left(0 + \frac{0.1}{2}\times 3\right) + 3\times e^{0.05}$$
$$= 3.453813289,$$

$$\zeta_0 = f\left(y_0 + \frac{h}{2}\eta_0, t_0 + \frac{h}{2}\right) = 2\left(y_0 + \frac{h}{2}\eta_0\right) + 3e^{\left(t_0+\frac{h}{2}\right)}$$
$$= 2\left(0 + \frac{0.1}{2}\times 3.453813289\right) + 3\times e^{0.05} = 3.499194618,$$
$$w_0 = f(y_0 + h\zeta_0, t_0 + h) = 2(y_0 + h\zeta_0) + 3e^{(t_0+h)}$$
$$= 2(0 + 0.1\times 3.499194618) + 3\times e^{0.1} = 4.015351678$$

Thus,

$$y_1 = y_0 + \frac{h}{6}[\xi_0 + 2\eta_0 + 2\zeta_0 + w_0]$$
$$= 0 + \frac{0.1}{6}(3 + 2\times 3.453813289 + 2\times 3.499194618 + 4.015351678) = 0.3486894582$$

For next iteration, we start from y_1, and calculate the values of ξ_1, η_1, ζ_1 and w_1. They are

$$\xi_1 = f(y_1, t_1) = 2y_1 + 3e^{t_1} = 2\times 0.3486894582 + 3\times e^{0.1} = 4.012891671,$$
$$\eta_1 = f\left(y_1 + \frac{h}{2}\xi_1, t_1 + \frac{h}{2}\right) = 2\left(y_1 + \frac{h}{2}\xi_1\right) + 3e^{\left(t_1+\frac{h}{2}\right)}$$
$$= 2\left(0.3486894582 + \frac{0.1}{2}\times 4.012891671\right) + 3\times e^{\left(0.1+\frac{0.1}{2}\right)} = 4.584170812,$$
$$\zeta_1 = f\left(y_1 + \frac{h}{2}\eta_1, t_1 + \frac{h}{2}\right) = 2\left(y_1 + \frac{h}{2}\eta_1\right) + 3\times e^{\left(t_1+\frac{h}{2}\right)}$$
$$= 2\left(0.3486894582 + \frac{0.1}{2}\times 4.584170812\right) + 3\times e^{\left(0.1+\frac{0.1}{2}\right)} = 4.641298726,$$
$$w_1 = f(y_1 + h\zeta_1, t_1 + h) = 2(y_1 + h\zeta_1) + 3\times e^{(t_1+h)}$$
$$= 2(0.3486894582 + 0.1\times 4.641298726) + 3\times e^{0.2} = 6.887058455$$

Thus,

$$y_2 = y_1 + \frac{h}{6}[\xi_1 + 2\eta_1 + 2\zeta_1 + w_1]$$
$$= 0.3486894582 + \frac{0.1}{6}\left(4.012891671 + 2\times 4.584170812 + 2\times 4.641298726 + 6.887058455\right)$$
$$= 0.83787094$$

Example 12: Use Runge-Kutta Gill method of fourth-order to solve the IVP $\dot{y} = 2y + 3e^t,\ y(0) = 0 \text{ using } \mathrm{h} = 0.1$. Carry out the solution for two time steps.

Solution: The first derivative is found from $\dot{y}_i = 2y_i + 3e^{t_i}$. To find y_i we must know $y_0, \xi_0, \eta_0, \zeta_0$ and w_0. They are

$$y_0 = 0,$$
$$\xi_0 = f\left(y_0, t_0\right) = 2y_0 + 3e^{t_0} = 2\times 0 + 3\times e^0 = 3,$$
$$\eta_0 = f\left(y_0 + \frac{h}{2}\xi_0, t_0 + \frac{h}{2}\right) = 2\left(y_0 + \frac{h}{2}\xi_0\right) + 3e^{\left(t_0 + \frac{h}{2}\right)}$$
$$= 2\left(0 + \frac{0.1}{2}\times 3\right) + 3\times e^{0.05} = 3.453813289,$$
$$\zeta_0 = f\left(y_0 + \frac{h}{\sqrt{2}}(\xi_0 - \eta_0) - \frac{h}{2}(\xi_0 - 2\eta_0), t_0 + \frac{h}{2}\right)$$
$$= 2\left(y_0 + \frac{h}{\sqrt{2}}(\xi_0 - \eta_0) - \frac{h}{2}(\xi_0 - 2\eta_0)\right) + 3e^{\left(t_0 + \frac{h}{2}\right)}$$
$$= 2\left(0 + \frac{0.1}{\sqrt{2}}\times -0.4538133 + \frac{0.1}{2}\times 3.9076266\right) + 3e^{\left(\frac{0.1}{2}\right)} = 3.480397056,$$
$$w_0 = f\left(y_0 - \frac{h}{\sqrt{2}}(\eta_0 - \zeta_0) + h\zeta_0, t_0 + h\right)$$
$$= 2\left(y_0 - \frac{h}{\sqrt{2}}(\eta_0 - \zeta_0) + h\zeta_0\right) + 3e^{(t_0 + h)} = 4.015351678$$

Thus,

$$y_1 = y_0 + \frac{h}{6}\left[\xi_0 + \left(2 - \sqrt{2}\right)\eta_0 + \left(2 + \sqrt{2}\right)\zeta_0 + w_0\right]$$
$$= 0 + \frac{0.1}{6}\left(3 + \left(2 - \sqrt{2}\right)\times 3.453813289 + \left(2 + \sqrt{2}\right)\times 3.480397056 + 4.015351678\right)$$
$$= 0.3486894582$$

For next iteration, we start from y_1, and calculate the values of ξ_1, η_1, ζ_1 and w_1. They are

$$\xi_1 = f(y_1, t_1) = 2y_1 + 3e^{t_1} = 4.012891671,$$

$$\eta_1 = f\left(y_1 + \frac{h}{2}\xi_1, t_1 + \frac{h}{2}\right) = 2\left(y_1 + \frac{h}{2}\xi_1\right) + 3e^{\left(t_1 + \frac{h}{2}\right)} = 4.584170812,$$

$$\zeta_1 = f\left(y_1 + \frac{h}{\sqrt{2}}(\xi_1 - \eta_1) - \frac{h}{2}(\xi_1 - 2\eta_1), t_1 + \frac{h}{2}\right)$$

$$= 2\left(y_1 + \frac{h}{\sqrt{2}}(\xi_1 - \eta_1) - \frac{h}{2}(\xi_1 - 2\eta_1)\right) + 3e^{\left(t_1 + \frac{h}{2}\right)} = 4.617635569,$$

$$w_1 = f\left(y_1 - \frac{h}{\sqrt{2}}(\eta_1 - \zeta_1) + h\zeta_1, t_1 + h\right)$$

$$= 2\left(y_1 - \frac{h}{\sqrt{2}}(\eta_1 - \zeta_1) + h\zeta_1\right) + 3e^{(t_1 + h)} = 5.289846936$$

Thus,

$$y_2 = y_1 + \frac{h}{6}\left[\xi_1 + \left(2 - \sqrt{2}\right)\eta_1 + \left(2 + \sqrt{2}\right)\zeta_1 + w_1\right] = 0.8112507529$$

Summary:

We now end this chapter by giving a summary of it. In this chapter we have covered the following

(1) The steps involved in solving the **IVP** $y' = f(y,t),\ y(\mathrm{t}_0) = \mathrm{y}_0, t \in [\mathrm{t}_0, b]$ by Euler's method are as follows :

Step 1: Evaluate $f(y_0, t_0)$

Step 2: Find $y_1 = y_0 + h\, f(y_0, t_0)$

Step 3: If $t_0 < b$, change $t_0\, to\, t_0 + h$ *and* y_0 *to* y_1 and repeat steps 1 and 2

Step 4: If $\mathrm{t}_0 = b$, write the value of y_1.

(2) Runge-Kutta methods being single step methods are self- starting methods.

(3) Unlike Taylor series methods, R-K methods do not need calculation of higher order derivatives f(y, t) but need only the evaluation of f(y, t) at the off-step points.

Exercise:

1. Use Euler’s method to Solve the IVP $\frac{dy}{dt}=4-2yt,\ y(0)=0.2. Find\ y(0.4)$ with $h=0.1$.

2. Solve the IVP $y'=\frac{1}{x^2-4y},\ y(4)=4,$ by Euler's method using $h=0.1.$ Carry out the solution for five time steps. Find the exact error.

3. Use Euler method to find the solution of $y'=t+|y|, given\ y(0)=1.$ Find the solution on [0, 0.8] with h = 0.2..

4. Solve the IVP $y'=1+y^2, y(0)=1$ and find $Find\ y(0.6)\ taking\ h=0.2\ and\ h=0.1$ using the Euler’s methods.

5. Solve the IVP $y'=-ty^2, y(2)=1$ and find $y(2.1)$ and $y(2.2)\ taking\ h=0.1$ using the following R-K methods of second order

a) Optimal R-K method
b) Improved tangent method
c) Heun's method 6. Solve the IVP $y'=3t+\frac{1}{2}y, y(0)=1$ and find $y(2.1)$ and $y(2.2)\ taking\ h=0.1$ using the following R-K methods of second order

a) Optimal R-K method
b) Improved tangent method
c) Heun's method

Compare the results with the exact solution $y(t)=13e^{t/2}-6t-12,$ and find the errors.

7. Use Optimal Runge-Kutta method of second-order to solve the IVP $10\,y' = y^2 + t^2\,,\, y(0.2) = 1,$ using $\mathrm{h} = 0.1.$

8. Use Classical Runge-Kutta method of fourth-order to solve the IVP $\dot{y} = y + t\,,\, y(0) = 1,$ using $\mathrm{h} = 0.1.$ Carry out the solution for five time steps. Also find the error at t = 0.5, if the exact solution is $y(t) = 2e^t - t - 1.$

9. Solve the IVP $y' = 2y + 3e^t\,,\, y(0) = 0$ and find $y(2.1)$ and $y(2.2)$ *taking* $h = 0.1$ using the following R-K methods of second order

a) Classical R-K method of fourth-order

b) R-K Gill method of fourth-order

10. Use Runge-Kutta Gill method of fourth-order to solve the IVP $\dfrac{dy}{dt} = 1 - 2\,y\,t\,,\; y(0.2) = 0.1948.\,Find\; y(0.2)$ with $h = 0.2$.

Glossary:

B Boundary conditions: n conditions which are prescribed at more than one point.

Boundary value problem (BVP): Differential equation together with the boundary.

C Classical Runge-Kutta Method of fourth:

$$y_{i+1} = y_i + \frac{h}{6}[\xi_i + 2\eta_i + 2\zeta_i + w_i]$$

where

$$\xi_i = f(y_i, t_i)$$

$$\eta_i = f\left(y_i + \frac{h}{2}\xi_i, t_i + \frac{h}{2}\right)$$

$$\zeta_i = f\left(y_i + \frac{h}{2}\eta_i, t_i + \frac{h}{2}\right)$$

$$w_i = f(y_i + h\zeta_i, t_i + h)$$

E Euler's Method:

$$y_{i+1} = h\dot{y}_i ,$$

where $h = \Delta t = t_{i+1} - t_i,$ is the step size.

H Henu's Method: The Runge-Kutta method of second order also known as the Euler-Cauchy method is

$$y_{i+1} = y_i + \frac{1}{2}(\xi_i + \eta_i)$$

where

$$\xi_i = h\, f\left(y_i, t_i\right)$$
$$\eta_i = h\, f\left(y_i + \xi_i\ , t_i + h\right)$$

I Improved Tangent Method: The Runge-Kutta method of second order also known as the modified Euler's method.

$$y_{i+1} = y_i + \eta_i$$

where

$$\xi_i = h\, f\left(y_i, t_i\right)$$
$$\eta_i = h\, f\left(y_i + \frac{\xi_i}{2}\ , t_i + \frac{h}{2}\right)$$

O Optimal Runge-Kutta Method: A second order Runge-Kutta method given by

$$y_{i+1} = y_i + \frac{1}{4}\left[\xi_i + 3\eta_i\ \right]$$

where

$$\xi_i = h\, f\left(y_i, t_i\right)$$
$$\eta_i = h\, f\left(y_i + \frac{2\xi_i}{3}\ , t_i + \frac{2h}{3}\right)$$

R Runge-Kutta Gill Method: A fourth order Runge-Kutta method given by

$$y_{i+1} = y_i + \frac{h}{6}\left[\xi_i + \left(2-\sqrt{2}\right)\eta_i + \left(2+\sqrt{2}\right)\zeta_i + w_i\right]$$

Where

$$\xi_i = f\left(y_i, t_i\right)$$
$$\eta_i = f\left(y_i + \frac{h}{2}\xi_i, t_i + \frac{h}{2}\right)$$
$$\zeta_i = f\left[y_i + \frac{h}{\sqrt{2}}(\xi_i - \eta_i) - \frac{h}{2}(\xi_i - 2\eta_i), t_i + \frac{h}{2}\right]$$
$$w_i = f\left[y_i - \frac{h}{\sqrt{2}}(\eta_i - \zeta_i) + h\zeta_i, t_i + h\right].$$

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